B.Sc Part I (Hons)

Fundamental theorem of Algebra (Problems)

Theorem:- Prove that in equation with real coefficients imaginary roots occur in conjugate pairs.

Proof:- Let $\alpha+i\beta$ be the root of the equation $f(x)=0$ in which the coefficients are real, then $\alpha-i\beta$ must also be a root of the equation $f(x)=0$.

When $\alpha+i\beta$ is a root of $f(x)=0$ then

$$f(\alpha+i\beta)=0 \quad \text{---------} \quad (1)$$

Not to follow division algorithm we divide the polynomial $f(x)$ by

$$[(x-(\alpha+i\beta)][x-(\alpha-i\beta)] \text{ i.e. by } (x-\alpha)^2+\beta^2.$$

if the quotient be Q and remainder be $Rx+R'$ then we have

$$f(x)=\{(x-\alpha)^2+\beta^2\}Q+(Rx+R') \quad \text{---------} \quad (i)$$

As $\alpha+i\beta$ is a root of equation then

$$f(\alpha+i\beta)=0, \text{ then by (1)}$$

$$f(\alpha+i\beta)=0=\{(\alpha+i\beta)-\alpha)^2+\beta^2\}Q+R(\alpha+i\beta)+R$$

$$\Rightarrow 0=R(\alpha+i\beta)+R'$$

$$\Rightarrow R\alpha+R'+Ri\beta=0+i0$$

Equating real and imaginary part, we get

$$R\alpha+R'=0 \text{ and } R\beta=0, \text{ i.e. } R=0 \text{ as } \beta\neq0$$

$$\therefore 0\alpha+R'=0 \Rightarrow R'=0$$

We have $R=0$ and $R'=0$, so by (i)

$$f(x) = \left[(x-\alpha)^2 + \beta^2\right] Q + 0 = (x-\alpha)^2 + \beta^2$$

$$= \left[(x-\alpha)^2 + (i\beta)^2\right] Q$$

$$f(x) = (x-\alpha-i\beta)(x-\alpha+i\beta) Q$$

This shows that $x-\alpha-i\beta$ is a factor of $f(x)$. i.e. $x = \alpha + i\beta$ in a root of $f(x) = 0$.

~~that~~ Proved:

Question:- From the equation with rational coefficients which shall have for two of its roots are $\sqrt{3}$ and $2+i$.

Ans:- We have when one of the roots of given equation is $\sqrt{3}$ then other must be $-\sqrt{3}$ and when one root is $2+i$ then other will be $2-i$.

The required equation will be

$$\{x-(2+i)\}\{x-(2-i)\}\{(x-\sqrt{3})(x+\sqrt{3})\} = 0$$

$$\{(x-2)-i\}\{(x-2)+i\}\{x^2-(\sqrt{3})^2\} = 0$$

$$\Rightarrow \{(x-2)^2 - i^2\}(x^2-3) = 0$$

$$\{(x-2)^2+1\}(x^2-3) = 0$$

$$(x^2+4-4x+1)(x^2-3) = 0$$

$$(\cancel{x^2+4x+}\, x^2-4x+5)(x^2-3) = 0$$

$$x^4 - 4x^3 + 2x^2 + 12x + 5 = 0$$

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